

R-POLYNOMIALS OF FINITE MONOIDS OF LIE TYPE

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ABSTRACT. This paper concerns the combinatorics of the orbit Hecke algebra associated with the orbit of a two sided Weyl group action on the Renner monoid of a finite monoid of Lie type, M . It is shown by Putcha in [12] that the Kazhdan-Lusztig involution ([6]) can be extended to the orbit Hecke algebra which enables one to define the R -polynomials of the intervals contained in a given orbit. Using the R -polynomials, we calculate the Möbius function of the Bruhat-Chevalley ordering on the orbits. Furthermore, we provide a necessary condition for an interval contained in a given orbit to be isomorphic to an interval in some Weyl group.

1. INTRODUCTION

Let $G = G(\mathbb{F}_q)$ be a finite group of Lie type (see [4]), $B \subseteq G$ a Borel subgroup, $T \subseteq B$ a maximal torus, W the Weyl group of T and S the set of simple reflections for W corresponding to B . Set

$$(1.1) \quad \epsilon = \frac{1}{|B|} \sum_{g \in B} g.$$

By a fundamental theorem of Tits, it is known that the algebra $\epsilon \mathbb{C}[G]\epsilon$ is isomorphic to the group algebra $\mathbb{C}[W]$.

The generic Hecke algebra $\mathcal{H}(W)$ of G , which is a deformation of the group algebra $\mathbb{C}[W]$, is a fundamental tool in combinatorics, geometry and the representation theory of G . As a $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ -algebra, $\mathcal{H}(W)$ is generated by a set of formal variables $\{T_w\}_{w \in W}$ indexed by the Weyl group W and obeys a corresponding multiplication rule.

In [20], Solomon introduces the first example of a Hecke algebra for monoids, in the case of the monoid $\mathbf{M}_n(\mathbb{F}_q)$, $n \times n$ matrices over a finite field \mathbb{F}_q . In a series of papers ([10], [12], [13]), Putcha extends the theory of Hecke algebras of matrices to all finite regular monoids. In particular, he defines the orbit Hecke algebra $\mathcal{H}(J)$ for a \mathcal{J} -class J in a finite regular monoid.

Let M be a finite monoid of Lie type and J a \mathcal{J} -class in M . Finite monoids of Lie type are regular monoids, and the \mathcal{J} -class $J = GeG$ in M is a $G \times G$ -orbit in M of the subgroup $G \subseteq M$ of invertible elements, where e is an idempotent of M . Here, G is a finite group of Lie type. (We use the notation $\mathcal{H}(e)$ in place of $\mathcal{H}(J)$.)

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The generic orbit Hecke algebra $\mathcal{H}(e)$ for GeG is what $\mathcal{H}(W)$ is for G . In other words, $\mathcal{H}(e)$ is a deformation of the contracted semigroup algebra $\epsilon \mathbb{C}_0[J^0]\epsilon$ (the zero of the algebra is the zero of $J^0 = J \cup \{0\}$), and ϵ is as in (1.1)).

In [12], Putcha extends the Kazhdan-Lusztig involution $\overline{T_w} = T_{w^{-1}}^{-1}$ to the generic orbit Hecke algebras. Using this involution, he defines analogues of R -polynomials and P -polynomials of [6]. In this article, we investigate the combinatorial properties of these R -polynomials. We show that given an interval contained in an orbit WeW inside the Renner monoid R of the monoid M ([16]), the constant term of the corresponding R -polynomial equals the value of the Möbius function on the given interval. Using this observation, we give a criterion for when a subinterval of WeW can be embedded into a Weyl group as a subinterval.

2. BACKGROUND

The monoids of Lie type are introduced and classified by Putcha in [9] and [11]. Among the important examples of these monoids are the finite reductive monoids, [17]. We begin with the notation of reductive monoids. For more information, interested readers may consult [18] and [8]. For an easy introduction on reductive monoids we especially recommend the exposé by L. Solomon [19].

Let K be an algebraically closed field. An algebraic monoid over K is an irreducible variety M such that the product map is a morphism of varieties. The set $G = G(M)$ of invertible elements of M is an algebraic group. If G is a reductive group, M is called a *reductive monoid*.

Let $B \subseteq G$ be a Borel subgroup, $T \subseteq B$ a maximal torus, $W = N_G(T)/T$ the Weyl group of the pair (G, T) and S the set of simple reflections for W corresponding to B , ℓ and \leq the length function and the Bruhat-Chevalley order corresponding to (W, S) .

Recall that the Bruhat-Chevalley decomposition

$$G = \bigsqcup_{w \in W} B\dot{w}B, \text{ for } w = \dot{w}T \in N_G(T)$$

of the reductive group G is controlled by the Weyl group W of G , where \dot{w} is any coset representative of $w \in W$.

In a reductive monoid M , the Weyl group W of the pair (G, T) and the set $E(\overline{T})$ of idempotents of the embedding $\overline{T} \hookrightarrow M$ form a finite inverse semigroup $R = \overline{N_G(T)}/T \cong W \cdot E(\overline{T})$ with the unit group W and the idempotent set $E(R) = E(\overline{T})$. The inverse semigroup R , called the *Renner monoid* of M , governs the Bruhat decomposition of the reductive monoid M :

$$M = \bigsqcup_{r \in R} BrB, \text{ for } r = \dot{r}T \in \overline{N_G(T)}.$$

Recall that the Bruhat-Chevalley order for (W, S) is defined by

$$x \leq y \text{ iff } BxB \subseteq \overline{ByB}.$$

Similarly, on the Renner monoid R of a reductive monoid M , the Bruhat-Chevalley order is defined by

$$(2.1) \quad \sigma \leq \tau \text{ iff } B\sigma B \subseteq \overline{B\tau B}.$$

Observe that the poset structure on W induced from R agrees with the original Bruhat poset structure on W .

Let $E(\overline{T})$ be the set of idempotent elements in the Zariski closure of the maximal torus T in the monoid M . Similarly, denote the set of idempotents in the monoid M by $E(M)$. One has $E(\overline{T}) \subseteq E(M)$. There is a canonical partial order \leq on $E(M)$ (hence on $E(\overline{T})$) defined by

$$(2.2) \quad e \leq f \Leftrightarrow ef = e = fe.$$

Note that $E(\overline{T})$ is invariant under the conjugation action of the Weyl group W . We call a subset $\Lambda \subseteq E(\overline{T})$ a *cross-section lattice* if Λ is a set of representatives for the W -orbits on $E(\overline{T})$ and the bijection $\Lambda \rightarrow G \backslash M / G$ defined by $e \mapsto GeG$ is order preserving. Then, $\Lambda = \Lambda(B) = \{e \in E(\overline{T}) : Be = eBe\}$.

The decomposition $M = \bigsqcup_{e \in \Lambda} GeG$, of a reductive monoid M into its $G \times G$ orbits, has a counterpart for the Renner monoid R of M . Namely, the finite monoid R can be written as a disjoint union

$$(2.3) \quad R = \bigsqcup_{e \in \Lambda} WeW$$

of $W \times W$ orbits, parametrized by the cross-section lattice Λ .

For $e \in \Lambda$, define

$$\begin{aligned} W(e) &:= \{x \in W : xe = ex\}, \\ W_e &:= \{x \in W : xe = e\} \trianglelefteq W(e). \end{aligned}$$

Both $W(e)$ and W_e are parabolic subgroups of W .

By $D(e)$ and D_e , denote the minimal coset representatives of $W(e)$ and W_e respectively:

$$\begin{aligned} D(e) &:= \{x \in W : x \text{ is of minimum length in } xW(e)\}, \\ D_e &:= \{x \in W : x \text{ is of minimum length in } xW_e\}. \end{aligned}$$

Any given element $\sigma \in WeW$ has the *standard form* xey^{-1} for unique x and y , where $x \in D_e$, and $y \in D(e)$ and $\sigma = xey^{-1}$.

The length function for R with respect to (W, S) is defined as follows:

Let w_0 and v_0 be the longest elements of W and $W(e)$ respectively. Then w_0v_0 is the longest element of $D(e)$. Set

$$\begin{aligned} \ell(e) &:= \ell(w_0v_0) = \ell(w_0) - \ell(v_0) \text{ and} \\ \ell(\sigma) &:= \ell(x) + \ell(e) - \ell(y). \end{aligned}$$

Note that ℓ , in general, need not be equal to the rank function on the graded poset (R, \leq) . However, when restricted to a \mathcal{J} -class $WeW \subseteq R$ it is equal to the rank function on the induced poset (WeW, \leq) .

By [7], we know that given two elements θ and σ in the standard form $\theta = uev^{-1} \in WeW$ and $\sigma = xfy^{-1} \in WfW$,

$$(2.4) \quad \theta \leq \sigma \iff e \leq f, u \leq xw, yw \leq v \quad \text{for some } w \in W(f)W_e.$$

Remark 2.1. More generally, let M be a finite monoid of Lie type, and $G \subseteq M$ its group of invertible elements. It is shown by Putcha, [9] that M has a Renner monoid R as well as a cross section lattice $\Lambda \subset E(M)$. Furthermore, all of what is said above is true for R and Λ of M .

3. ORBIT HECKE ALGEBRAS

Let M be a finite monoid of Lie type. We use the notation of the previous section.

In [20], Solomon constructs the Hecke algebra and the generic Hecke algebra for the monoid $M_n(\mathbb{F}_q)$, $n \times n$ matrices over the finite field \mathbb{F}_q with q elements.

Until the end of the section, we let q be an indeterminate instead of a prime power. Following Solomon's construction in [20], the generic Hecke algebra $\mathcal{H}(R)$ of the Renner monoid of M is defined as follows:

The generic Hecke algebra $\mathcal{H}(R)$ is a $\mathbb{Z}[q^{-\frac{1}{2}}, q^{\frac{1}{2}}]$ -algebra generated by a formal basis $\{A_\sigma\}_{\sigma \in R}$ with respect to multiplication rules

$$(3.1) \quad A_s A_\sigma = \begin{cases} A_\sigma & \text{if } \ell(s\sigma) = \ell(\sigma) \\ A_{s\sigma} & \text{if } \ell(s\sigma) = \ell(\sigma) + 1 \\ q^{-1} A_{s\sigma} + (1 - q^{-1}) A_\sigma & \text{if } \ell(s\sigma) = \ell(\sigma) - 1 \end{cases}$$

$$A_\nu A_\sigma = A_{\nu\sigma}$$

for $s \in S$, $\sigma, \nu \in R$, where $\ell(\nu) = 0$. The products $A_\sigma A_s$ are defined similarly.

Fix $e \in \Lambda$ and let $\mathcal{I} \subseteq \mathcal{H}(R)$ be the two-sided ideal

$$(3.2) \quad \mathcal{I} = \bigoplus_{f < e, \sigma \in W f W} \mathbb{Z}[q^{-\frac{1}{2}}, q^{\frac{1}{2}}] A_\sigma \subseteq \mathcal{H}(R).$$

The *orbit Hecke algebra*

$$(3.3) \quad \mathcal{H}(e) = \bigoplus_{\sigma \in W e W} \mathbb{Z}[q^{-\frac{1}{2}}, q^{\frac{1}{2}}] A_\sigma$$

is an ideal of $\mathcal{H}(R)/\mathcal{I}$. Note that, if the idempotent $e \in \Lambda$ is the identity element $\text{id} \in W$ of the Weyl group, then the generic orbit Hecke algebra $\mathcal{H}(e)$ is isomorphic to generic Hecke algebra $\mathcal{H}(W)$ of G .

The algebra $\widehat{\mathcal{H}}(e) := \mathcal{H}(e) + \mathcal{H}(W)$ is called the *augmented orbit Hecke algebra*.

Theorem 3.1 ([12], Theorem 4.1). *There is a unique extension of the involution on $\mathcal{H}(W)$ to $\widehat{\mathcal{H}}(e)$ such that for $e \in W e W$ and $\sigma \in W e W$ in standard form $\sigma = s e t^{-1}$,*

$$\begin{aligned} \overline{A}_e &:= \sum_{z \in W(e), y \in D(e)} \overline{R}_{z,y} A_{zey^{-1}}, \\ \overline{A}_\sigma &:= q^{-\ell(t)} \overline{A}_s \sum_{z \in W(e), y \in D(e)} \overline{R}_{tz,y} A_{zey^{-1}}. \end{aligned}$$

Here $R_{z,y}, R_{tz,y} \in \mathbb{Z}[q]$ are R -polynomials of W .

Corollary 3.2 ([12], Corollary 4.2). *Let $\sigma \in W e W$. Then there exists $R_{\theta,\sigma} \in \mathbb{Z}[q]$ for $\theta \in W e W$, such that in $\widehat{\mathcal{H}}(e)$,*

- (i) $\overline{A}_\sigma = q^{\ell(\sigma)-\ell(e)} \sum_{\theta \in W e W} \overline{R}_{\theta,\sigma} A_\theta$,
- (ii) $R_{\theta,\sigma} \neq 0$ only if $\theta \leq \sigma$,
- (iii) $R_{\theta,\theta} = 1$.

In Section 5, we answer the following question by Putcha:

Problem 3.3 ([12], Problem 4.3.). *Determine the polynomials $R_{\theta,\sigma}$ explicitly for $\theta, \sigma \in W e W$. Does $\theta \leq \sigma$ imply $R_{\theta,\sigma} \neq 0$?*

4. DESCENT SETS FOR ELEMENTS OF THE RENNER MONOID

An important ingredient in the study of the combinatorics of the Kazhdan-Lusztig theory for Weyl groups is the descent of an element $w \in W$, which has been missing in the context of Renner monoids. In the following, we extend the notion of the descent set of an element $w \in W$ to a \mathcal{J} -class (a $W \times W$ -orbit) in the Renner monoid.

Note that for a simple reflection s and $\theta \in R$,

$$s\theta < \theta \text{ (resp., } =, >) \quad \text{if and only if} \quad \ell(s\theta) - \ell(\theta) = -1 \text{ (resp., } 0, 1).$$

The following lemma can be found in [14].

Lemma 4.1. *Let $I \subset S$, W_I the subgroup generated by I , and D_I the minimal coset representatives of W/W_I in W . Given $x, y \in D_I$ and $w, u \in W_I$.*

- (i) *If $xw < yu$ then there exist $w_1, w_2 \in W$ satisfying $w = w_1 w_2$ such that $\ell(w) = \ell(w_1) + \ell(w_2)$, $xw_1 \leq y$ and $w_2 \leq u$.*
- (ii) *If $wx^{-1} < uy^{-1}$ then there exist $w_1, w_2 \in W$ satisfying $w = w_1 w_2$ such that $\ell(w) = \ell(w_1) + \ell(w_2)$, $w_1 \leq u$ and $w_2 x^{-1} \leq y^{-1}$.*

Corollary 4.2. *We use the notation of the previous Lemma. Let $x \in D_I$ and let $s \in S$.*

- (i) *If $x < sx$ then either $sx \in D_I$ or $sx = xs'$ for some $s' \in W_I \cap S$.*
- (ii) *If $sx < x$ then $sx \in D_I$.*

Proof. (i) Suppose $sx = x's'$ for some $x' \in D_I$ and $s' \in W_I$. Let id denote the identity element of the Weyl group. We have

$$x \cdot \text{id} = x(\text{id} \cdot \text{id}) \leq x's'$$

and by previous Lemma it follows that $x \leq x'$ and therefore $\ell(x) \leq \ell(x')$. On the other hand since

$$\ell(x) + 1 = \ell(sx) = \ell(x's') = \ell(x') + \ell(s')$$

we have either $x = x'$ and $s' \in I$ or $s' = \text{id}$ and $x' = sx$.

(ii) If W is finite and $W_I \subseteq W$ then Björner and Wachs shows in [3, Theorem 4.1] that D_I , which is a generalized quotient, is a lower interval of the weak Bruhat order of W . Therefore the result follows. \square

For $\sigma \in W e W$, define the left descent set and the right descent of σ with respect to S as

$$\text{Des}_L(\sigma) = \{s \in S \mid \ell(s\sigma) < \ell(\sigma)\} \text{ and } \text{Des}_R(\sigma) = \{s \in S \mid \ell(\sigma s) < \ell(\sigma)\}.$$

Then, by the Corollary 4.2 we reformulate $\text{Des}_L(\sigma)$ and $\text{Des}_R(\sigma)$ as follows:

Lemma 4.3. Suppose $\sigma \in WeW$ has the standard form xey^{-1} where $x \in D_e$ and $y \in D(e)$. Then $\text{Des}_L(\sigma) = \{s \in S \mid \ell(sx) < \ell(x)\}$, and $\text{Des}_R(\sigma) = \{s \in S \mid \ell(sy) > \ell(y)\}$, and either $sy \in D(e)$, or $sy = ys'$ for some $s' \in W(e) \cap S$ and $\ell(xs') < \ell(x)$.

Remark 4.4. Let $\nu \in WeW$ be the unique element satisfying $\ell(\nu) = 0$. Then, it is easy to see that both descent sets of ν are empty. It is essential to emphasize that unlike the usual Weyl group setting, not every $\sigma \in WeW$ has a left or right descent. On the other hand, by using [3, Theorem 4.1], one can show the following.

Corollary 4.5. For $\sigma \in WeW$ such that $\ell(\sigma) \neq 0$ we have $\text{Des}_L(\sigma) \cup \text{Des}_R(\sigma) \neq \emptyset$.

The following example illustrates the possible cases for the descent sets of $\sigma \in WeW$ for $W = S_n$.

Example 4.6. Let $M_4(\mathbb{F}_q)$ be the finite monoid of 4×4 matrices over the finite field \mathbb{F}_q with q elements. The Renner monoid of $M_4(\mathbb{F}_q)$ consists of all 4×4 partial permutation matrices¹, and its Weyl subgroup is the symmetric group $W = S_4$ consisting of permutation matrices. Given a matrix $x = (x_{ij})$ in the Renner monoid, let $(a_1 a_2 a_3 a_4)$ be the sequence defined by

$$(4.1) \quad a_j = \begin{cases} 0, & \text{if the } j\text{th column consists of zeros;} \\ i, & \text{if } x_{ij} = 1. \end{cases}$$

For example, the sequence associated with the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

is (3040) . Let e be the idempotent $e = (1200) \in WeW$. Then, $W(e) \cong S_2 \times S_2$. The table below illustrates the possible cases for the descent sets for some $\sigma \in WeW$

$\underline{\sigma}$	$\underline{\text{Des}}_L$	$\underline{\text{Des}}_R$
$(1234)e(3412) = (0012)$	\emptyset	\emptyset
$(1324)e(3412) = (0013)$	$\{s_2\}$	\emptyset
$(1234)e(1342) = (1002)$	\emptyset	$\{s_1\}$
$(3214)e(1342) = (3002)$	$\{s_1, s_2\}$	$\{s_1\}$
$(4213)e(3124) = (0420)$	$\{s_1, s_3\}$	$\{s_2, s_3\}$

Here, $s_1 = (2134)$, $s_2 = (1324)$, $s_3 = (1243)$ are the simple reflections for S_4 .

¹A partial permutation matrix is a $0 - 1$ matrix with at most one 1 in each row and each column.

5. R-POLYNOMIALS

Given an interval $[\theta, \sigma] \subseteq WeW$ define its length $\ell(\theta, \sigma) := \ell(\sigma) - \ell(\theta)$ and its R -polynomial $R_{\theta, \sigma}(q)$ as in Corollary 3.2.

Theorem 5.1. *Let $\sigma, \theta \in R$ be such that $\ell(\sigma) \neq 0$ and $\theta \leq \sigma$. Then for $s \in \text{Des}_L(\sigma)$, one has*

$$R_{\theta, \sigma} = \begin{cases} R_{s\theta, s\sigma} & \text{if } s\theta < \theta, \\ qR_{\theta, s\sigma} & \text{if } s\theta = \theta, \\ (q-1)R_{\theta, s\sigma} + qR_{s\theta, s\sigma} & \text{if } s\theta > \theta. \end{cases}$$

Otherwise there exists $s \in \text{Des}_R(\sigma)$ and

$$R_{\theta, \sigma} = \begin{cases} R_{\theta s, \sigma s} & \text{if } \theta s < \theta, \\ qR_{\theta, \sigma s} & \text{if } \theta s = \theta, \\ (q-1)R_{\theta, \sigma s} + qR_{\theta s, \sigma s} & \text{if } \theta s > \theta. \end{cases}$$

An addition the above, if $s\theta > \theta$ and $s\sigma = \sigma$, then $R_{\theta, \sigma} = qR_{s\theta, \sigma}$.

When $\theta = \sigma$, $R_{\theta, \sigma}(q) = 1$. If $[\theta, \sigma]$ is an interval of length 1 in WeW , then $R_{\theta, \sigma}(q) = q - 1$.

Remark 5.2. *Given two elements u, v of a Weyl group W , the polynomial $R_{u, v}(q) \neq 0$ if and only if $u \leq v$. If $u \leq v$, then $R_{u, v}(q)$ is a monic polynomial of degree $\ell(u, v)$ whose constant term is $(-1)^{\ell(u, v)}$.*

For the orbit Hecke algebras, we have the following which answers Problem 3.3 [12]:

Proposition 5.3. *Let $\theta \leq \sigma \in WeW$. Then $R_{\theta, \sigma}$ is a monic polynomial of degree $\ell(\theta, \sigma) = \ell(\sigma) - \ell(\theta)$ whose the constant term is either 0 or $(-1)^{\ell(\theta, \sigma)}$. In particular,*

$$R_{\theta, \sigma} \neq 0 \quad \text{if and only if} \quad \theta \leq \sigma.$$

Proof. We prove the statement about the constant term. The statement about the degree and the leading term is proved similarly via induction on $\ell(\sigma)$.

Clearly, the constant term statement holds if $\ell(\sigma) \leq 1$. As the induction hypothesis, we assume that for all pairs $\rho \leq \tau$ in WeW with $\ell(\tau) < \ell(\sigma)$, the constant term of $R_{\rho, \tau}(q)$ is either 0 or $(-1)^{\ell(\rho, \tau)}$.

Let $\theta \in WeW$ be such that $\theta \leq \sigma$. If $R_{\theta, \sigma}(0) = 0$, there is nothing to prove.

Assume that the constant term of $R_{\theta, \sigma}$ is non-zero. Without loss of generality we assume that there exists $s \in \text{Des}_L(\sigma)$. Then, by Theorem 5.1, we must have either $s\theta > \theta$ or $s\theta < \theta$.

First suppose that $s\theta < \theta$. Then $R_{\theta, \sigma}(q) = R_{s\theta, s\sigma}(q)$. Since $\ell(s\theta, s\sigma) = \ell(\theta, \sigma)$, $R_{\theta, \sigma}(0)$ equals $(-1)^{\ell(\theta, \sigma)}$.

Now suppose that $s\theta > \theta$. Therefore, $R_{\theta, \sigma} = (q-1)R_{\theta, s\sigma} + qR_{s\theta, s\sigma}$. Note that $\theta \leq s\sigma$. Consequently, $R_{\theta, \sigma}(0) = -R_{\theta, s\sigma}(0)$. Hence the latter is nonzero, and by the induction hypothesis it equals $(-1)^{\ell(\theta, s\sigma)}$. Therefore, $R_{\theta, \sigma}(0) = (-1)^{\ell(\theta, \sigma)}$ as claimed. \square

Remark 5.4. *A very similar line of argument shows that if $R_{\theta, \sigma}(q)$ has a non-zero constant term, then $\overline{R_{\theta, \sigma}} = \varepsilon_\theta \varepsilon_\sigma q_\theta q_\sigma^{-1} R_{\theta, \sigma}(q)$. However, this equality is false if $R_{\theta, \sigma}(0) = 0$.*

The lifting property for Weyl group W states that given $u < v$ in W and a simple reflection s , if $u > su$ and $v < sv$, then $u < sv$ and $su < v$. We will use the lifting property for W to prove the lifting property for the orbits WeW .

Corollary 5.5 (Lifting Property for WeW). *Let $\theta = uev^{-1}$ and $\sigma = xey^{-1}$ be in standard form, $\theta < \sigma$ and s be a simple reflection.*

- (a) *If $\theta < s\theta$ and $\sigma < s\sigma$, then $s\theta < s\sigma$.*
- (b) *If $s\theta \geq \theta$ and $s\sigma \leq \sigma$, then $\theta \leq s\sigma$ and $s\theta \leq s\sigma$.*

Proof. To make matters short, use Theorem 5.1 and Proposition 5.3 to prove (a) or (b) if any of the inequalities is an equality, .

What remains to be shown is (b) in the strict case: $s\theta > \theta$ and $s\sigma < \sigma$.

Because $s\sigma < \sigma$, by Lemma 4.3, $sx < x$ and by Corollary 4.2 (b), $sx \in D_e$. Thus, the standard form for $s\sigma$ is $(sx)ey^{-1}$.

Since $s\theta > \theta$, we observe that $su \not\leq u$. Otherwise, $(su)ev^{-1}$ is the standard form of $s\theta$ resulting in a contradiction $\ell(s\theta) < \ell(\theta)$.

As $\theta \leq \sigma$, there is $w \in W(e)$ so that $u \leq xw$ and $yw \leq v$. First, we prove $\theta \leq s\sigma$.

- Case $xw \leq s(xw)$: Then $u \leq xw \leq sxw$. Because $(sx)ey^{-1}$ is the standard form of $s\sigma$, we conclude that $\theta \leq s\sigma$.
- Case $s(xw) \leq xw$: Apply the lifting property for W to u and xw . So, $u \leq s(xw) = (sx)w$ and again, we get $\theta \leq s\sigma$.

To prove $s\theta \leq \sigma$, apply (a) to the pair $\theta < s\sigma$.

□

Let q_σ and ε_σ denote, respectively, $q^{\ell(\sigma)}$, and $(-1)^{\ell(\sigma)}$ for $\sigma \in WeW$.

Proposition 5.6. *For all $\theta, \sigma \in WeW$,*

$$(5.1) \quad \sum_{\theta \leq \nu \leq \sigma} R_{\theta, \nu} q_\sigma q_\nu^{-1} \overline{R_{\nu, \sigma}} = \delta_{\theta, \sigma}.$$

We call an interval $[\theta, \sigma]$ linear if the interval $[\theta, \sigma]$ is totally ordered with respect to Bruhat-Chevalley order. In this case, the interval $[\theta, \sigma]$ has $\ell(\theta, \sigma) + 1$ elements.

Using the above proposition together with Proposition 5.3, one can classify length 2 intervals $[\theta, \sigma] \subset WeW$ with respect to their R -polynomials or equivalently, with respect to the constant terms of their R -polynomials:

(5.2)

Shape of Bruhat Graph	Number of Elements	Example: $[\theta, \sigma] \subset \mathbf{M}_4(\mathbb{F}_q)$	$R_{\theta, \sigma}(q)$	$R_{\theta, \sigma}(0)$
linear	3	$(0001) < (0003)$	$q(q-1)$	0
diamond	4	$(0012) < (0023)$	$(q-1)^2$	1

Length 2 intervals will play a very crucial role later on.

The Möbius function $\mu_{\theta,\sigma}$ corresponding to the interval $[\theta, \sigma]$ is defined by

$$(5.3) \quad \mu_{\theta,\sigma} := \begin{cases} 0 & \text{if } \theta \not\leq \sigma, \\ 1 & \text{if } \theta = \sigma, \\ -\sum_{\theta \leq \tau < \sigma} \mu_{\theta,\tau} & \text{if } \theta < \sigma. \end{cases}$$

Corollary 5.7. *For $\theta, \sigma \in WeW$,*

$$\mu_{\theta,\sigma} = R_{\theta,\sigma}(0).$$

Proof. The equality is clear for $\theta \not\leq \sigma$ and $\theta = \sigma$. For $\theta < \sigma$, evaluating (5.1) at $q = 0$ yields

$$R_{\theta,\sigma}(0) = - \sum_{\theta \leq \tau < \sigma} R_{\theta,\tau}(0).$$

Thus, $\mu_{\theta,\sigma} = R_{\theta,\sigma}(0)$. □

Putcha conjectures (Conjecture 2.7 [15]) the following for reductive monoids, a subclass of monoids of Lie type:

$$\mu_{\theta,\sigma} = \begin{cases} (-1)^{\ell(\sigma)-\ell(\theta)} & \text{if every length 2 interval in } [\theta, \sigma] \text{ has 4 elements,} \\ 0 & \text{otherwise.} \end{cases}$$

We prove that

Theorem 5.8. *Putcha's conjecture holds for the \mathcal{J} -classes in the monoids of Lie type.*

To prove this theorem, we examine the interplay between the R -polynomial $R_{\theta,\sigma}$, the interval it belongs to $[\theta, \sigma]$ and the subintervals $[\alpha, \beta]$ contained in $[\theta, \sigma]$, esp. of length 2 subintervals of $[\theta, \sigma]$.

Next, we prove a relation between $R_{\theta,\sigma}(0)$ and the $R_{\alpha,\beta}(0)$ for a subinterval $[\alpha, \beta]$ of $[\theta, \sigma]$.

Proposition 5.9. *Given an interval $[\theta, \sigma]$ with $R_{\theta,\sigma}(0) \neq 0$, then*

- (a) *If $s\sigma < \sigma$ (or $s\theta > \theta$, $\sigma s < \sigma$, $\theta s > \theta$) for some simple reflection s , then for any $\alpha \in [\theta, \sigma]$, $s\alpha \neq \alpha$.*
- (b) *For a subinterval $[\alpha, \beta]$ of $[\theta, \sigma]$ in WeW , $R_{\alpha,\beta}(0) \neq 0$.*

In the proof, we will prove (a) for the case $s\sigma < \sigma$, other cases being virtually the same.

Proof. Prove by induction on $\ell(\sigma)$. If $\ell(\sigma) \leq 1$, then $R_{\theta,\sigma}(q) = (q-1)^{\ell(\theta,\sigma)}$ and both statements hold trivially.

Induction step. Note that $s\theta \neq \theta$. If $s\theta < \theta$, then one can apply induction to $[s\theta, s\sigma]$ as $R_{s\theta,s\sigma}(0) = R_{\theta,\sigma}(0) \neq 0$. If $s\theta > \theta$, then one can apply induction to $[\theta, s\sigma]$ as $R_{\theta,s\sigma}(0) = -R_{\theta,\sigma}(0) \neq 0$.

Let $\rho := \min\{\theta, s\theta\}$. By definition, $s\rho > \rho$ and by lifting property, for any $\alpha \in [\theta, \sigma]$, $\rho \leq \min\{\alpha, s\alpha\}$. In short, we can apply induction to $[\rho, s\sigma]$ since $\ell(s\sigma) < \ell(\sigma)$.

(a) $R_{\theta,\sigma}(0) \neq 0$ and a simple reflection s so that $s\sigma < \sigma$ is given. If for all $\alpha \in [\theta, \sigma]$, $s\alpha \neq 0$, there is nothing to prove.

Assume that for some $\alpha \in [\theta, \sigma]$, $s\alpha = \alpha$. By lifting property, $\alpha \leq s\sigma$ and hence $\alpha \in [\rho, s\sigma]$. Use induction to conclude that $s\alpha \neq \alpha$ as $s\rho > \rho$. This gives a contradiction. Therefore, we conclude that there is no $\alpha \in [\theta, \sigma]$ such that $s\alpha = \alpha$.

(b) Assume that $s\sigma < \sigma$ for some $s \in S$.

Pick any interval $[\alpha, \beta]$ in $[\theta, \sigma]$. If $s\beta > \beta$, by the lifting property $[\alpha, \beta] \subset [\rho, s\sigma]$ (apply the induction step).

Otherwise, $s\beta < \beta$.

There are two cases:

- $s\alpha < \alpha$: Then, $R_{\alpha, \beta}(0) = R_{s\alpha, s\beta}(0)$ and $[s\alpha, s\beta] \subset [\rho, s\sigma]$ (apply the induction step).
- $s\alpha > \alpha$: Then, $R_{\alpha, \beta}(0) = -R_{\alpha, s\beta}(0)$ and $[\alpha, s\beta] \subset [\rho, s\sigma]$ (apply the induction step).

□

One might wonder if for all *proper* subintervals $[\alpha, \beta] \subset [\theta, \sigma]$, then $R_{\alpha, \beta}(0) \neq 0$, is it true that $R_{\theta, \sigma}(0) \neq 0$? Unfortunately, the answer is no. As a counterexample, take any linear length 2 interval $[\theta, \sigma]$. Then, any proper subinterval $[\alpha, \beta]$ is of length ≤ 1 , hence $R_{\alpha, \beta}(0) \neq 0$, yet $R_{\theta, \sigma}(0) = 0$.

6. INTERVALS OF LENGTH ≤ 2

It is clear that an interval $[\theta, \sigma]$ in WeW with $R_{\theta, \sigma}(0) = 0$ can never be embedded into some Weyl group W' as a subinterval $[u, v]$ so that the $R_{\theta, \sigma}(q)$ equals $R_{u, v}(q)$ for the simple reason that $R_{u, v}(q) = (-1)^{\ell(u, v)} \neq 0$. In fact, more is true. We prove the following fact about the Bruhat graphs of the intervals $[\theta, \sigma]$ with $R_{\theta, \sigma}(0) = 0$:

Theorem 6.1. *An interval $[\theta, \sigma]$ in WeW cannot be embedded into any Weyl group W' as a subinterval if $R_{\theta, \sigma}(0) = 0$.*

We prove this assertion by showing that

Proposition 6.2. *Given an interval $[\theta, \sigma]$ in WeW , $R_{\theta, \sigma}(0) = 0$ iff there exists a linear length 2 interval $[\alpha, \beta]$ inside $[\theta, \sigma]$.*

Theorem 6.1 follows from this proposition because of the well-known fact that any interval $[u, v]$ of length 2 in a Weyl group W' is diamond shaped and contains 4 elements.

Lemma 6.3. *Given an interval $[\theta, \sigma]$ in WeW , if there exists a simple reflection $s \in S$ such that $s\sigma < \sigma$ (or, $s\theta > \theta$, $s\sigma > \sigma$, $s\theta > \theta$) and $s\rho = \rho$ (resp. $s\rho = \rho$) for some $\rho \in [\theta, \sigma]$, then $[\theta, \sigma]$ contains a linear length 2 interval.*

Proof. We prove the lemma for $s\sigma < \sigma$ as all other cases are essentially proved the same way.

By assumption, the set $\{\rho : s\rho = \rho\}$ is non-empty. Pick a maximal element α in this set. Then, for all $\beta \in [\theta, \sigma]$ with $\alpha < \beta$, $s\beta \neq \beta$.

Now pick $\beta \in [\alpha, \sigma]$ so that it covers α . By the choice of α , $s\beta \neq \beta$ as indicated above. Because β covers α , $R_{\alpha, \beta}(q) = q - 1$.

If it were that $s\beta < \beta$, then by the recurrence relations, Theorem 5.1, $q - 1 = qR_{\alpha, s\beta}(q)$, which implies that $\alpha = s\beta$ and $q - 1 = q$, both being obvious contradictions.

Therefore, $s\beta > \beta$ and $R_{\alpha,s\beta}(0) = 0$. By lifting property, $s\beta \leq \sigma$. The subinterval $[\alpha, s\beta]$ is a linear length 2 interval in $[\theta, \sigma]$ as required. \square

Lemma 6.4. *Let $[\theta, \sigma]$ be an interval which contains a linear length 2 interval $[\alpha, \beta]$. Say for some $s \in S$, $\theta < s\theta$ and $\sigma < s\sigma$. Then, the interval $[s\theta, s\sigma]$ contains a linear length 2 interval.*

Proof. Say $s\alpha \leq \alpha$. By lifting property, $s\theta \leq \alpha < \beta < \sigma < s\sigma$. The interval $[\alpha, \beta] \subset [s\theta, s\sigma]$ is the required one.

Otherwise, $s\alpha > \alpha$. Since $[\alpha, \beta]$ is a linear length 2 interval, $s\beta \geq \beta$. If $\beta > s\beta$, then $R_{s\alpha,s\beta} = R_{\alpha,\beta}$ and $s\theta \leq s\alpha < s\beta \leq s\sigma$.

If $s\beta = \beta$, then by lifting property $s\theta \leq \beta \leq \sigma < s\sigma$. The results follows by Lemma 6.3. \square

Proof of Proposition 6.2. (\Leftarrow) If there is such α, β , then $R_{\alpha,\beta}(0) = 0$ and the rest follows from Proposition 5.9.

(\Rightarrow) Prove by induction on the length $\ell(\sigma)$.

The base case is $\ell(\theta) = 0$ and $\ell(\sigma) = 2$ which follows from (5.2). As usual, assume that $s\sigma < \sigma$ for a simple reflection $s \in S$.

If $s\theta = \theta$, the result follows by Lemma 6.3.

If $s\theta > \theta$, then $R_{\theta,\sigma} = (q - 1)R_{\theta,s\sigma} + qR_{s\theta,s\sigma}$. Hence $0 = R_{\theta,\sigma}(0) = R_{\theta,s\sigma}(0)$. The length of the interval $[\theta, s\sigma]$ is $\ell(\theta, \sigma) - 1$. Apply induction.

If $s\theta < \theta$, then apply the Lemma above and then induction. This ends the proof of the Proposition.

Proof of Theorem 5.8. We reiterate what we have already shown.

At this point, we have showed the following are equivalent: For a given interval $[\theta, \sigma]$ in $W \backslash W$,

- (1) $\mu_{\theta,\sigma} = R_{\theta,\sigma}(0) \neq 0$,
- (2) $\mu_{\theta,\sigma} = R_{\theta,\sigma}(0) = (-1)^{\ell(\theta,\sigma)}$,
- (3) All length 2 subintervals of $[\theta, \sigma]$ have 4 elements (and are diamond shaped).

Otherwise, $\mu_{\theta,\sigma} = R_{\theta,\sigma}(0) = 0$ and $[\theta, \sigma]$ contains a length 2 subinterval with 3 elements (a linear length 2 subinterval).

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